

## JOINT UNIVERSALITY FOR LERCH ZETA-FUNCTIONS

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ABSTRACT. For  $0 < \alpha, \lambda \leq 1$ , the Lerch zeta-function is defined by  $L(s; \alpha, \lambda) := \sum_{n=0}^{\infty} e^{2\pi i \lambda n} (n + \alpha)^{-s}$ , where  $\sigma > 1$ . In this paper, we prove joint universality for Lerch zeta-functions with distinct  $\lambda_1, \dots, \lambda_m$  and transcendental  $\alpha$ .

## 1. INTRODUCTION AND STATEMENT OF MAIN RESULT

For  $0 < \alpha, \lambda \leq 1$ , we define the Lerch zeta-function by

$$L(s; \alpha, \lambda) := \sum_{n=0}^{\infty} \frac{e(\lambda n)}{(n + \alpha)^s}, \quad \sigma > 1,$$

where  $e(t) = \exp(2\pi it)$ . When  $\lambda = 1$ , the function  $L(s; \alpha, \lambda)$  reduces to the Hurwitz zeta-function  $\zeta(s, \alpha)$ . If  $\lambda \neq 1$ , the Lerch zeta-function  $L(s; \alpha, \lambda)$  is analytically continuable to an entire function. However, the Hurwitz zeta-function  $\zeta(s, \alpha)$  is extended to a meromorphic function, which has a simple pole at  $s = 1$ .

In this paper, we show the following joint universality theorem expected by Mishou [6, Conjecture 1]. In order to state it, put  $D := \{s \in \mathbb{C} : 1/2 < \operatorname{Re} s < 1\}$  and let  $\operatorname{meas}\{A\}$  be the Lebesgue measure on  $\mathbb{R}$  of the set  $A$ .

**Theorem 1.** *Suppose that  $L(s; \alpha, \lambda_1), \dots, L(s; \alpha, \lambda_m)$  are Lerch zeta-functions with distinct  $\lambda_1, \dots, \lambda_m$  and transcendental  $\alpha$ . For  $1 \leq j \leq m$ , let  $K_j \subset D$  be compact sets with connected complements and  $f_j(s)$  be continuous function on  $K_j$  and analytic in the interior of  $K_j$ . Then, for every  $\varepsilon > 0$ , we have*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \max_{1 \leq j \leq m} \max_{s \in K_j} |L(s + i\tau; \alpha, \lambda_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Roughly speaking, this theorem implies that any analytic functions can be simultaneously and uniformly approximated by Lerch zeta-functions with distinct  $\lambda_1, \dots, \lambda_m$ . The proof will be written in Sections 2 and 3. We skip the detail of the proofs of results appeared in Section 2 since they do not contain essentially new ideas. In Section 3, we prove the denseness lemma using an orthogonality of Dirichlet coefficients of the zeta-functions. The main idea of our proof was recently observed in [5] by the authors. However, in the present paper we adopt this approach to completely different kind of zeta-functions without Euler product. It proves the conjecture on joint universality for Lerch zeta-functions put forward by Mishou in [6] and shows that this idea can be applicable to many collections of zeta

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and  $L$ -functions, which independence relies on some orthogonality property of their coefficients.

Now we look back in the history of the joint universality for Lerch zeta-functions. Laurinćikas showed Theorem 1 with  $m = 1$  in [2, Theorem] (see also [3, Theorem 6.1.1]). Laurinćikas and Matsumoto proved Theorem 1 with the condition that  $\lambda_j = k_j/l_j$  are distinct rational numbers satisfying  $(k_j, l_j) = 1$  and  $0 < k_j \leq l_j$  in [4, Theorem 1] (see also [3, Theorem 6.3.1] or [6, Theorem 2]). In [7, Theorem 17], Nakamura obtained the joint universality of the Lerch zeta-functions with  $\lambda_j = \lambda + k_j/l_j$ , where  $0 < \lambda \leq 1$  and  $\lambda_j$  are distinct in mod 1. We have to mention that the method in the both papers [4, Theorem 1] and [7, Theorem 17] are based on the observation that if  $e(\lambda_l n) \neq e(\lambda_k n)$ , there exists  $M \in \mathbb{N}$  such that for all  $1 \leq k \neq l \leq m$ ,

$$|e(\lambda_l n) - e(\lambda_k n)| = |1 - e((\lambda_l - \lambda_k)n)| \geq |1 - e(1/M)|.$$

Recently, Mishou proved in [6, Theorem 4], the joint universality of the Lerch zeta-functions for almost all real numbers  $\lambda_j$ ,  $1 \leq j \leq m$  such that  $1, \lambda_1, \dots, \lambda_m$  are linearly independent over  $\mathbb{Q}$ . His proof is based on some results of discrepancy estimate from uniform distribution theory (see [6, Section 2]). Obviously, Theorem 1 of the present paper is not only an improvement of Mishou's result [6, Theorem 4] but also the final answer to [6, Conjecture 1].

By using Theorem 1, we get the following corollaries. We omit their proofs since they follow from the standard argument (see for example [3, Section 7.2]).

**Corollary 2.** *Let  $\alpha \in (0, 1]$  be transcendental and  $\lambda_1, \dots, \lambda_m \in (0, 1]$  be distinct real numbers. For  $N \in \mathbb{N}$  and  $1/2 < \sigma < 1$ , define the mapping  $h: \mathbb{R} \rightarrow \mathbb{C}^{mN}$  by the formula*

$$h(t) := (L(\sigma + i\tau; \alpha, \lambda_1), L'(\sigma + i\tau; \alpha, \lambda_1), \dots, L^{(N-1)}(\sigma + i\tau; \alpha, \lambda_1), \\ \dots, L(\sigma + i\tau; \alpha, \lambda_m), L'(\sigma + i\tau; \alpha, \lambda_m), \dots, L^{(N-1)}(\sigma + i\tau; \alpha, \lambda_m)).$$

*Then the image of  $\mathbb{R}$  is dense in  $\mathbb{C}^{mN}$ .*

**Corollary 3.** *Let  $\alpha \in (0, 1]$  be transcendental and  $\lambda_1, \dots, \lambda_m \in (0, 1]$  be distinct real numbers. Suppose  $N \in \mathbb{N}$  and  $F_l$ ,  $0 \leq l \leq k$  are continuous functions on  $\mathbb{C}^{mN}$  and satisfy*

$$\sum_{l=0}^k s^l F_l(L(s; \alpha, \lambda_1), L'(s; \alpha, \lambda_1), \dots, L^{(N-1)}(s; \alpha, \lambda_1), \\ \dots, L(s; \alpha, \lambda_m), L'(s; \alpha, \lambda_m), \dots, L^{(N-1)}(s; \alpha, \lambda_m)) \equiv 0.$$

*Then we have  $F_l \equiv 0$  for  $0 \leq l \leq k$ .*

## 2. PROOF OF THEOREM 1

Recall that  $D := \{s \in \mathbb{C} : 1/2 < \operatorname{Re} s < 1\}$  and denote by  $H(D)$  the space of analytic function on  $D$  equipped with the topology of uniform convergence on compacta. Let  $\mathfrak{B}(X)$  stand for the class of Borel sets of the space  $X$ . Define by  $\gamma$  the unit circle on  $\mathbb{C}$ , and let  $\Omega := \prod_{n=0}^{\infty} \gamma_n$ , where  $\gamma_n = \gamma$  for all  $n \in \mathbb{N}_0$ . Denoting by  $m_H$  the probability Haar measure on  $(\Omega, \mathfrak{B}(\Omega))$ , we obtain a probability space

$(\Omega, \mathfrak{B}(\Omega), m_H)$ . For  $\sigma > 1$ , we define

$$L(s; \alpha, \lambda; \omega) := \sum_{n=0}^{\infty} \frac{e(\lambda n) \omega(n)}{(n + \alpha)^s}, \quad \omega(n) \in \gamma.$$

Note that for almost all  $\omega \in \Omega$  the series above converges uniformly on compact subsets of  $D$  (see for instance [3, Lemma 5.2.1]).

Let  $H(D)^m := H(D) \times \cdots \times H(D)$ . We define a probability measure  $P_T$  on  $(H(D)^m, \mathfrak{B}(H(D)^m))$  by

$$P_T(A) := \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : (L(s + i\tau; \alpha, \lambda_1), \dots, L(s + i\tau; \alpha, \lambda_m)) \in A \right\},$$

where  $A \in \mathfrak{B}(H(D)^m)$ . Next define the  $H(D)^m$ -valued random element  $\underline{L}(s; \omega)$  by

$$\underline{L}(s; \omega) := (L(s; \alpha, \lambda_1; \omega), \dots, L(s; \alpha, \lambda_m; \omega)).$$

Denote by  $P_{\underline{L}}$  the distribution of the random element  $\underline{L}(s; \omega)$ , namely,

$$P_{\underline{L}}(A) := m_H \{ \omega \in \Omega : \underline{L}(s; \omega) \in A \}, \quad A \in \mathfrak{B}(H(D)^m).$$

Then we have the following limit theorem proved by Matsumoto and Laurinćikas [4] (see also [3, Theorem 5.3.1] or [6, Section 5]).

**Proposition 4** ([4, Lemma 1]). *Let  $0 < \alpha < 1$  be transcendental. Then the probability measure  $P_T$  converges weakly to  $P_{\underline{L}}$  as  $T \rightarrow \infty$ .*

The proof of the next lemma shall be written in Section 3 since it contains the most novel part of the present paper.

**Lemma 5.** *The set  $\{\underline{L}(s; \omega) : \omega \in \Omega\}$  is dense in  $H(D)^m$ .*

Recall that the minimal closed set  $S_{\mathbf{P}} \subset X$  such that  $\mathbf{P}(S_{\mathbf{P}}) = 1$  is called the support of a probability space  $(X, \mathfrak{B}(X), \mathbf{P})$ . The set  $S_{\mathbf{P}}$  consists of all  $x \in S$  such that for every neighborhood  $V$  of  $x$  the inequality  $\mathbf{P}(V) > 0$  is satisfied. From Lemma 5 and [3, Lemma 6.1.3] or [9, Lemma 12.7], the support of the probability measure  $P_{\underline{L}}$  is  $H(D)^m$ . First assume that  $h_1(s), \dots, h_m(s) \in H(D)$  are polynomials. Let  $K_j$  be the same as in Theorem 1 and  $\Phi$  be the set of functions  $\varphi \in H(D)^m$  which satisfy

$$\max_{1 \leq j \leq m} \max_{s \in K_j} |\varphi_j(s) - h_j(s)| < \varepsilon.$$

From Proposition 4, the definition of support, Portmanteau theorem (see for instance [9, Theorem 3.1]) and the fact that the support of  $P_{\underline{L}}$  is  $H(D)^m$ , we have

$$\liminf_{T \rightarrow \infty} P_T(\Phi) \geq P_{\underline{L}}(\Phi) > 0.$$

Therefore, we obtain

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{1 \leq j \leq m} \max_{s \in K_j} |L(s + i\tau; \alpha, \lambda_j) - h_j(s)| < \varepsilon \right\} > 0.$$

Hence it suffices to show that polynomials  $h_j(s)$  can be replaced by  $f_j(s)$  appeared in Theorem 1. It is possible by Mergelyan's theorem which implies that any function  $f(s)$  which is continuous on  $K$  and analytic in the interior of  $K$ , where  $K$  is a compact subset with connected complement, is uniformly approximative on  $K$  by polynomials. Hence we omit the details since this is easily done by the well-known method (see for example [3, p. 129] or [6, p. 1125]).

## 3. PROOF OF LEMMA 5

Let  $U$  be a simply connected smooth Jordan domain such that  $\overline{U} \subset D$ . Let  $B^2(U)$  be the Bergman space of all holomorphic square integrable complex functions with respect to the Lebesgue measure on  $U$  with the inner product

$$\langle f, g \rangle = \iint_U f(s) \overline{g(s)} d\sigma dt, \quad f, g \in H(U).$$

The properties below are well-known (see for instance [8]).

**Lemma 6** ([8, Proposition 7.2.2 and Theorem 7.2.3]). *We have the following.*

- (a) *Convergence in  $B^2(U)$  implies local uniform convergence on  $U$ .*
- (b)  *$B^2(U)$  is a Hilbert space.*
- (c) *The set of polynomials is dense in  $B^2(U)$ .*

Now let  $\mathbb{B}^m := B^2(U) \times \cdots \times B^2(U)$  is the Hilbert space with the inner product given, for  $\underline{f} = (f_1, \dots, f_m) \in H(U)^m$  and  $\underline{g} = (g_1, \dots, g_m) \in H(U)^m$  by

$$\langle \underline{f}, \underline{g} \rangle = \sum_{j=1}^m \iint_U f_j(s) \overline{g_j(s)} d\sigma dt.$$

In order to prove Lemma 5, we use (b) of Lemma 6 and the following result appeared, for example, in [9].

**Lemma 7** ([9, Theorem 6.1.16]). *Let  $H$  be a complex Hilbert space. Assume that a sequence  $v_n \in H$ ,  $n \in \mathbb{N}$  satisfies*

- (i) *the series  $\sum_n \|v_n\|^2 < \infty$ ;*
  - (ii) *for any element  $0 \neq g \in H$  the series  $\sum_n |\langle v_n, g \rangle|$  is divergent.*
- Then the set of convergent series*

$$\left\{ \sum_n a_n v_n \in H : |a_n| = 1 \right\}$$

*is dense in  $H$ .*

Let  $\underline{g} = (g_1, \dots, g_m) \in \mathbb{B}^m$  be a non-zero element and put

$$\underline{v_n}(s) := (v_n(s; \alpha, \lambda_1), \dots, v_n(s; \alpha, \lambda_m)), \quad v_n(s; \alpha, \lambda_j) := \frac{e(\lambda_j n)}{(n + \alpha)^s}.$$

Then for  $\Delta_j(w) := \iint_U e^{-sw} \overline{g_j(s)} d\sigma dt$ , one has

$$\langle \underline{v_n}(s), \underline{g}(s) \rangle = \sum_{j=1}^m e(\lambda_j n) \Delta_j(\log(n + \alpha)).$$

We can see that the condition (i) of Lemma 7 is true since  $\overline{U} \subset D$  and

$$\langle \underline{v_n}(s), \underline{v_n}(s) \rangle = \sum_{j=1}^m \iint_U (n + \alpha)^{-s} \overline{(n + \alpha)^{-s}} d\sigma dt \ll \sup_{s \in U} |(n + \alpha)^{-2s}|.$$

The truth of the condition (ii) in Lemma 7 easily follows from the following crucial lemma.

**Lemma 8.** *Assume that  $\underline{g}(s) = (g_1(s), \dots, g_m(s)) \in \mathbb{B}^m$  is a non-zero element and for  $j = 1, \dots, m$ , put  $\Delta_j(z) := \iint_U e^{-sz} \overline{g_j(s)} d\sigma dt$ . Then the following series*

$$\sum_{n=0}^{\infty} |e(\lambda_1 n) \Delta_1(\log(n + \alpha)) + \dots + e(\lambda_m n) \Delta_m(\log(n + \alpha))|$$

*is divergent.*

In order to prove the lemma above, we quote the following.

**Lemma 9** ([5, Corollary 2.7]). *Let  $\|g_j\| \neq 0$  for  $1 \leq j \leq m$ . Then for every  $A > 0$  and every  $x > 1$ , there exist sequences  $B_1 > \dots > B_m > 0$ ,  $x_0^{(0)} = x, x_0^{(1)}, \dots, x_0^{(m)}$  and intervals  $I_j \subset [x, x + 1]$  of length  $|I_j| \geq B_j x^{-2j}$  such that  $x_0^{(j)} \in I_j$ ,  $I_{j+1} \subset I_j$ , and for all  $t \in I_j$  we have*

$$(1) \quad \begin{aligned} \frac{1}{2} |\Delta_j(x_0^{(j-1)})| + O(e^{-Ax}) &\leq \frac{1}{2} |\Delta_j(x_0^{(j)})| + O(e^{-Ax}) \\ &\leq |\Delta_j(t)| \leq |\Delta_j(x_0^{(j)})| + O(e^{-Ax}). \end{aligned}$$

*Proof of Lemma 8.* Without loss of generality, we can assume that  $g_1$  is a non-zero element since  $\|\underline{g}\| \neq 0$  implies that at least one of  $g_j$ 's is a non-zero element.

Obviously,  $\Delta_1(z) \ll e^{C|z|}$  for some positive constant  $C$  depending on  $U$ . Let  $\sigma_1$  and  $\sigma_2$  be real numbers with  $1/2 < \sigma_1 < \sigma_2 < 1$  such that the vertical strip  $\sigma_1 < \operatorname{Re} s < \sigma_2$  contains the simply connected smooth Jordan domain  $U$ . Then for sufficiently small  $\eta = \eta(U) > 0$  and for all complex  $z$  with  $|\arg(-z)| \leq \eta$  we have  $|e^{\sigma_2 z} \Delta_1(z)| \ll 1$  from the definitions of  $U$  and  $\sigma_2$ . Furthermore,  $\Delta_1$  is not identically zero since otherwise for every  $k \in \mathbb{N}$  we have

$$0 = \Delta_1^{(k)}(0) = \iint_U (-s)^k \overline{g_1(s)} d\sigma dt,$$

which implies that  $g_1$  is orthogonal to all polynomials in  $B^2(U)$ , however, it contradicts to (c) of Lemma 6 and the assumption that  $\|g_1\| \neq 0$ . Hence, by using [1, Lemma 3], we can find a real sequence  $x_k$  tending to infinity such that

$$|\Delta_1(x_k)| \gg e^{-\sigma_2 x_k}.$$

Fix  $k$  and put  $x = x_k$ . Hence, by using Corollary 9, we can see that for every  $A > 0$  and  $x = x_k$ , there exist sequences  $B_1 > \dots > B_m > 0$ ,  $x_0^{(0)} = x, x_0^{(1)}, \dots, x_0^{(m)}$  and intervals  $I_j \subset [x, x + 1]$  of length  $|I_j| \geq B_j x^{-2j}$  such that  $x_0^{(j)} \in I_j$ ,  $I_{j+1} \subset I_j$ , and for all  $t \in I_j$ , the inequalities (1) holds. Now let  $I_m := [y, y + B_m y^{-2m}] \subset [x, x + 1]$ . Since  $I_m \subset I_j$  for every  $j = 1, 2, \dots, m$ , the inequalities (1) holds also for all  $t \in I_m$ . In particular, since  $x_0^{(0)} = x$ , for  $t \in I_m$  one has

$$(2) \quad |\Delta_1(t)| \geq \frac{1}{2} |\Delta_1(x_0^{(0)})| \gg e^{-\sigma_2 x}.$$

Moreover, for every  $j = 1, 2, \dots, m$  we have

$$(3) \quad |\Delta_j(t)| \ll e^{-\sigma_1 x}, \quad t \in [x, x + 1].$$

We denote by  $\sum_n^*$  the sum over integers  $n + \alpha \in [e^y, e^{y+B_m y^{-2m}}]$  in order to obtain  $\log(n + \alpha) \in I_m$ .

First we consider the following sum

$$S_1(x) := \sum_n^* \sum_{j=1}^m |\Delta_j(\log(n + \alpha))|^2.$$

Obviously, it holds that

$$e^{y+y^{-2m}} - e^y = e^y (e^{y^{-2m}} - 1) = \frac{e^y}{y^{2m}} \sum_{n=0}^{\infty} y^{-2mn} \gg \frac{e^y}{y^{2m}}.$$

Let  $A > 0$  be sufficiently large. Then by using (1), (2),  $x \leq y \leq x + 1$  and the formula above, we have

$$\begin{aligned} S_1(x) &\gg \sum_n^* \sum_{j=1}^m \left( |\Delta_j(x_0^j)|^2 + |\Delta_j(x_0^j)| O(e^{-Ax}) + O(e^{-2Ax}) \right) \\ &\gg \sum_n^* \sum_{j=1}^m \left( |\Delta_j(x_0^j)|^2 + O(e^{-Ax}) \right) \gg \sum_n^* \left( \sum_{j=1}^m |\Delta_j(x_0^j)| \right)^2 \\ &\gg \sum_n^* e^{-\sigma_2 x} \sum_{j=1}^m |\Delta_j(x_0^j)| \gg \frac{e^{x(1-\sigma_2)}}{x^{2m}} \sum_{j=1}^m |\Delta_j(x_0^j)|. \end{aligned}$$

Since  $\lambda_k \neq \lambda_l$  for  $k \neq l$  from the assumption of Theorem 1, then it is easy to prove that for any  $1 \leq k \neq l \leq m$ , one has

$$\phi_{k,l}(t) := \sum_{n \leq t} e((\lambda_k - \lambda_l)n) \ll 1.$$

Similarly to (3), one can easily get the estimation

$$\frac{d}{du} \Delta_j(\log u) = \frac{1}{u} \Delta_j'(\log u) \ll u^{-1-\sigma_1}.$$

From  $\overline{\Delta_j(\log u)} = \overline{\langle u^{-s}, g_j(s) \rangle} = \langle u^{-\bar{s}}, \overline{g_j(s)} \rangle$ , we obtain

$$\frac{d}{du} \overline{\Delta_j(\log u)} = \frac{1}{u} \int \int_U -\bar{s} u^{-\bar{s}} \overline{g_j(s)} d\sigma dt = \frac{1}{u} \overline{\Delta_j'(\log u)} \ll u^{-1-\sigma_1}.$$

Hence, using partial summation, we have

$$\begin{aligned} &\sum_{X_1 \leq n \leq X_2} \sum_{1 \leq k \neq l \leq m} e((\lambda_k - \lambda_l)n) \Delta_k(\log(n + \alpha)) \overline{\Delta_l(\log(n + \alpha))} \\ &= \sum_{1 \leq k \neq l \leq m} \int_{X_1}^{X_2} \Delta_k(\log(u + \alpha)) \overline{\Delta_l(\log(u + \alpha))} d\phi_{k,l}(u) \\ &\ll X_1^{-2\sigma_1} + \sum_{1 \leq k \neq l \leq m} \int_{X_1}^{X_2} \left| \left( \Delta_k(\log(u + \alpha)) \overline{\Delta_l(\log(u + \alpha))} \right)' \right| du \\ &\ll X_1^{-2\sigma_1} + \int_{X_1}^{X_2} \frac{du}{u^{1+2\sigma_1}} \ll X_1^{-2\sigma_1} \end{aligned}$$

for sufficiently large  $X_2 > X_1 > 0$ . Thus we obtain

$$S_2(x) := \sum_{1 \leq k \neq l \leq m} \sum_n^* e((\lambda_l - \lambda_k)n) \Delta_k(\log(n + \alpha)) \overline{\Delta_l(\log(n + \alpha))} \ll e^{-2\sigma_1 x}.$$

We can easily see that

$$\begin{aligned} S(x) &:= \sum_n^* \left| e(\lambda_1 n) \Delta_1(\log(n + \alpha)) + \cdots + e(\lambda_m n) \Delta_m(\log(n + \alpha)) \right|^2 \\ &= S_1(x) + S_2(x) \gg \frac{e^{x(1-\sigma_2)}}{x^{2m}} \sum_{j=1}^m |\Delta_j(x_0^j)| + O(e^{-2\sigma_1 x}) \end{aligned}$$

when  $A$  is sufficiently large. On the other hand, one has

$$\begin{aligned} S(x) &\ll \sum_n^* \left| \sum_{j=1}^m e(\lambda_j n) \Delta_j(\log(n + \alpha)) \right| \left| \sum_{j=1}^m |\Delta_j(\log(n + \alpha))| \right| \\ &\ll \sum_n^* \left| \sum_{j=1}^m e(\lambda_j n) \Delta_j(\log(n + \alpha)) \right| \left| \sum_{j=1}^m |\Delta_j(x_0^{(j)})| \right| + O(e^{-(A+\sigma_1-1)x}). \end{aligned}$$

Hence, dividing the last inequalities by  $\sum_{j=1}^m |\Delta_j(x_0^{(j)})|$ , we have

$$\sum_n^* \left| \sum_{j=1}^m e(\lambda_j n) \Delta_j(\log(n + \alpha)) \right| \gg \frac{e^{x(1-\sigma_2)}}{x^{2m}},$$

since  $2\sigma_1 - \sigma_2 > 0$ . Thus, the last inequality implies Lemma 8.  $\square$

*Proof of Lemma 5.* We put

$$\begin{aligned} v_n(s, \omega(n); \alpha, \lambda_j) &:= \frac{e(\lambda_j n) \omega(n)}{(n + \alpha)^s}, \quad \omega(n) \in \gamma, \\ \underline{v}_n(s, \omega(n)) &:= (v_n(s, \omega(n); \alpha, \lambda_1), \dots, v_n(s, \omega(n); \alpha, \lambda_m)). \end{aligned}$$

Recall  $U$  be a simply connected smooth Jordan domain such that  $\overline{U} \subset D$ . Then the set of convergent series

$$\left\{ \sum_n \underline{v}_n(s, \omega(n)) : \omega \in \Omega \right\}$$

is dense in the space  $\mathbb{B}^m$  by Lemmas 7 and 8. Now we extend this result to the space  $H(U)^m$ . Let  $\varepsilon > 0$  and  $(h_1(s), \dots, h_m(s)) \in H(U)^m$ . From Lemma 8 there exists a sequence  $\beta(n) \in \gamma$  such that  $\sum_{n=0}^{\infty} \underline{v}_n(s, \beta(n))$  converges on  $U$  in the topology of  $\mathbb{B}^m$ . This convergence is uniform on every compact subsets  $\mathcal{K}_1, \dots, \mathcal{K}_m \subset U$  by (a) of Lemma 6 (see also [6, Lemma 7]). Thus we can find  $b(n) \in \gamma$  and  $M \in \mathbb{N}$  satisfying

$$\begin{aligned} \max_{1 \leq j \leq m} \max_{s \in \mathcal{K}_j} \left| \sum_{n=0}^M v_n(s, b(n); \alpha, \lambda_j) - h_j(s) \right| &< \frac{\varepsilon}{2}, \\ \max_{1 \leq j \leq m} \max_{s \in \mathcal{K}_j} \left| \sum_{n>M} v_n(s, b(n); \alpha, \lambda_j) \right| &< \frac{\varepsilon}{2} \end{aligned}$$

from (a) of Lemma 6 and Lemma 8. The inequality above and the assumption  $\overline{U} \subset D$  implies Lemma 5.  $\square$

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